

POLYHARMONIC APPROXIMATION ON THE SPHERE

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ABSTRACT. The purpose of this article is to provide new error estimates for a popular type of SBF approximation on the sphere: approximating by linear combinations of Green's functions of polyharmonic differential operators. We show that the L_p approximation order for this kind of approximation is σ for functions having L_p smoothness σ (for σ up to the order of the underlying differential operator, just as in univariate spline theory). This improves previous error estimates, which penalized the approximation order when measuring error in L_p , $p > 2$ and held only in a restrictive setting when measuring error in L_p , $p < 2$.

1. INTRODUCTION

Spherical basis functions (or SBFs) have been used with much success in multivariate approximation theory, statistics and a multitude of other scientific disciplines. At the heart of the SBF methodology is the creation of an approximant

$$s_{\Xi}(x) = \sum_{\xi \in \Xi} A_{\xi} \mathbf{k}(x \cdot \xi)$$

by taking a linear combination of rotations of a fixed kernel $(x, \alpha) \mapsto \mathbf{k}(x \cdot \alpha)$ (known as an SBF, or, sometimes, a *zonal* kernel).

The success of the SBF methodology derives from its ability to generate approximants from data having arbitrary geometry – a desirable quality on spheres, where geometry of data is always essentially unstructured: for an arbitrary spacing, there are no regular distributions of points on the sphere, meaning that approximation techniques requiring grids, regular triangulations, or other geometrical props do not work in this setting. Interpolation [4], [3], [9] and [7] and other SBF approximation methods [11], [12], [18], [10] (see bibliography in [3] for even more examples), are both frequently used to fit scattered data on the sphere.

Our focus is not on how to treat spherical data, but how to approximate smooth functions using SBF approximants, having access to as much information about the target function as necessary. The choice of coefficients $(A_{\xi})_{\xi \in \Xi}$ is a crucial element in the performance of the approximation, but,

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at the outset, we are not focused on a specific method of choosing coefficients. Instead, we wish to investigate the approximation power of this methodology for a robust family – the polyharmonic kernels (see Definition 3.1) – rather than any specific implementation or algorithm; the main concern is to establish accurate error analysis for approximation from spaces of polyharmonic SBFs, $S(\mathbf{k}, \Xi) := \text{span}_{\xi \in \Xi} \mathbf{k}(\cdot, \xi) + \Pi$, where a low dimensional space of elementary functions, Π , may be added to the span of the SBF.

The method for gauging the approximation power is the L_p *approximation order*, which measures the decay of the error in approximating from $S(\mathbf{k}, \Xi)$ as Ξ becomes dense in \mathbb{S}^d . For target functions f from a class \mathcal{F} , the approximation order is the largest exponent s so that

$$\|f - s_{f, \Xi}\|_{L_p(\mathbb{S}^d)} = \mathcal{O}(h^s)$$

where h , the ‘fill distance’, measures the density of Ξ in \mathbb{S}^d (see the following section for a precise definition of fill distance). In this setting, the rate is given in terms of the density of the centers Ξ , and depends strongly on the class \mathcal{F} of target functions.

When approximants are chosen from a predetermined linear space, independent of the target function, as is the case here (in contrast to *nonlinear approximation*, where the set of centers Ξ could be chosen independent of f), precise approximation theory ties the L_p approximation order to the L_p smoothness of the target function, e.g., by measuring the error in terms of an L_p modulus of smoothness or by selecting target functions in an L_p Sobolev or Besov space, $\mathcal{F} = W_p^s$ or $B_{p,q}^s$.

The prevailing method for estimating error for SBF (and, more generally, kernel) approximation has been to assume the target function resides in a reproducing kernel Hilbert space, often called the *native space*, for which the SBF acts as the reproducing kernel. Quite often, the native space is actually an L_2 Sobolev space. One drawback of this approach has been that it precludes finding faster rates for functions with more smoothness, or slower rates for less smoothness. Another drawback is that the L_p approximation orders degrade as p aberrates from 2 – see, e.g., [6] for an example of this criticism for ‘radial basis functions’ (or RBFs) in domains in \mathbb{R}^2 . We remark that [10, Corollary 3.5], and [9, Corollary 3 (a)] are examples of this phenomenon, but we place special emphasis on the results of Hubbert and Morton [8, Theorem 3.4, 3.8], because their results are the current state-of-the-art for the setting of this article. We paraphrase their result.

Theorem (Hubbert, Morton). *For an SBF \mathbf{k} having native space $W_2^m(\mathbb{S}^d)$, and for sufficiently dense centers Ξ , if $f \in W_2^m(\mathbb{S}^d)$ then the SBF interpolant s_f satisfies:*

$$\|f - s_f\|_{L_p(\mathbb{S}^d)} = \mathcal{O}\left(h^{m - (\frac{d}{2} - \frac{d}{p})_+}\right)$$

If $f \in W_2^{2m}(\mathbb{S}^d)$, s_f satisfies:

$$\|f - s_f\|_{L_p(\mathbb{S}^d)} = \mathcal{O}\left(h^{2m - (\frac{d}{2} - \frac{d}{p})_+}\right)$$

When $p > 2$, each approximation order is penalized by subtracting a positive term: $\frac{d}{2} - \frac{d}{p}$; when $p < 2$, the space of target functions is an L_2 Sobolev space of the form $W_2^\sigma(\mathbb{S}^d)$ which is embedded in $W_p^\sigma(\mathbb{S}^d)$. This should be contrasted with M^{th} order univariate spline approximation, which provides L_p approximation order σ for functions in $W_p^\sigma(\mathbb{S}^d)$ for a range of $0 < \sigma \leq M$, where M is the ‘saturation’ order – the rate beyond which any increase in smoothness fails to produce an increased rate of convergence. Our main results show (in Theorem 6.1 and its corollaries) for a polyharmonic kernel, \mathbf{k} , satisfying the conditions of the above theorem, and for a target function f having smoothness $\sigma \leq 2m$ in L_p there is $s_{f,\Xi} \in S(\mathbf{k}, \Xi)$ so that $\|f - s_{f,\Xi}\|_p = \mathcal{O}(h^\sigma)$.

In this paper, we develop an approximation scheme delivering novel error estimates for a robust family of SBFs: the ‘polyharmonic’ kernels. This is the family of Green’s functions of iterated and perturbed Laplace–Beltrami operators (see Definition 3.1 for a precise definition). Such kernels have been studied by Freeden and his collaborators, cf. [3] and references therein. They include the Green’s functions for Δ^m , and, thus, are direct generalizations of the periodic “Bernoulli splines” (famously studied in [4]) and are, in some sense, the spherical analogues of the “surface splines” used in \mathbb{R}^d . On the other hand, the SBFs obtained by directly restricting the \mathbb{R}^{d+1} surface splines to \mathbb{S}^d are, perhaps surprisingly, often represented in this family.

The scheme developed in this article is based on replacing the kernel in an integral identity by a linear combination of (few) scattered rotations of the kernel. This method has recently been introduced by DeVore and Ron in [2] where it was used to obtain nonlinear and local results for RBF approximation in the boundary-free, Euclidean setting. Later, it was used in [6], [5] to provide precise approximation orders for RBF approximation in domains in \mathbb{R}^2 .

The layout of this article is as follows. In Section 2 we discuss some basics of analysis on spheres. Section 3 introduces the kernels used in this paper and shows that they can be expressed as a sum of surface splines. In Section 4 we establish a basic strategy for exchanging the kernel by a linear combination of its copies. Section 5 estimates the error in making this exchange, while Section 6 collects our main results.

2. BACKGROUND

We denote by \mathbb{S}^d the unit sphere in \mathbb{R}^{d+1} , and by ω_d we denote its volume. The distance between two points, x and α , on the sphere is written $\text{dist}(x, \alpha) := \arccos(x \cdot \alpha)$. The basic neighborhood is the spherical ‘cap’ $C(\alpha, \rho) := \{x \in \mathbb{S}^d : \text{dist}(x, \alpha) < \rho\}$. Throughout this article, Ξ is assumed to be a finite subset of \mathbb{S}^d , and the ‘fill distance’,

$$h := h(\Xi, \mathbb{S}^d) := \max_{\alpha \in \mathbb{S}^d} \text{dist}(\alpha, \Xi),$$

measures the density of Ξ in \mathbb{S}^d

The spherical harmonic, as studied in [13], is the basic tool of Fourier analysis on the sphere. For each eigenvalue, $\nu_\ell := \ell(\ell + d - 1)$ of the Laplace – Beltrami operator Δ on \mathbb{S}^d , there corresponds an eigenspace of ‘spherical harmonics’ of exact degree ℓ , called \mathcal{H}_ℓ , having dimension $N(d, \ell) := \frac{(2\ell+d-1)\Gamma(\ell+d-1)}{\Gamma(\ell+1)\Gamma(d)}$ with orthonormal (in the sense of L_2) basis $(Y_{m,\ell})_{m=1}^{N(d,\ell)}$. The space of spherical harmonics of degree less than or equal to L is denoted $\Pi_L = \sum_{\ell \leq L} \mathcal{H}_\ell$.

In this article, our focus is on *zonal kernels*. These are kernels on the sphere having the form $(x, \alpha) \mapsto \phi(x \cdot \alpha)$, where $\phi : [-1, 1] \rightarrow \mathbb{R}$. Such kernels, being the composition of an inner product with a univariate function, can be expressed in terms of an expansion in orthogonal polynomials. The Gegenbauer (or ultraspherical) polynomials, $(P_\ell^{(\lambda)})_{\ell=0}^\infty$, are orthogonal on $[-1, 1]$ with respect to the weight $(1 - t^2)^{\lambda-1/2}$. We expand zonal functions on \mathbb{S}^d using $(P_\ell^{(\lambda_d)})_{\ell=0}^\infty$, with $\lambda_d := \frac{d-1}{2}$. Gegenbauer coefficients are

$$a_\ell := \int_{-1}^1 \phi(t) P_\ell^{(\lambda_d)}(t) (1 - t^2)^{(d-2)/2} dt$$

and the expansion is $\phi(x \cdot \alpha) = \sum_{\ell=0}^\infty a_\ell P_\ell^{(\lambda_d)}(x \cdot \alpha)$. This can be expressed, via the addition theorem for spherical harmonics [13, Theorem 2], as:

$$\phi(x \cdot \alpha) = \sum_{\ell=0}^\infty \frac{\lambda_d + \ell}{\omega_d \lambda_d} \widehat{\phi}(\ell) P_\ell^{(\lambda_d)}(x \cdot \alpha) = \sum_{\ell=0}^\infty \sum_{m=1}^{N(d,\ell)} \widehat{\phi}(\ell) Y_{\ell,m}(x) Y_{\ell,m}(\alpha)$$

where we supplant the Gegenbauer coefficient a_ℓ by the Fourier coefficient $\widehat{\phi}(\ell) := \frac{\omega_d \lambda_d}{\lambda_d + \ell} a_\ell$. We note that the polynomials used by Müller, [13], which he calls Legendre polynomials and denotes by \mathcal{P}_ℓ (suppressing the dependence on d), are normalized in L_∞ : they satisfy $\|\mathcal{P}_\ell\|_{L_\infty[-1,1]} = \mathcal{P}_\ell(1) = 1$. The Gegenbauer polynomials used here are normalized in $L_2([-1, 1]; (1 - t^2)^{\lambda-1/2})$, and are related to Müller’s Legendre polynomials by: $P_\ell^{(\lambda_d)} = \binom{\ell+2\lambda_d-1}{\ell} \mathcal{P}_\ell$. Basics of Gegenbauer polynomials can be found in [15, Section 4.7]. A key result relates the smoothness of the kernel ϕ with the decay of its Fourier coefficients, $\widehat{\phi}(\ell)$.

Proposition 2.1. *If $\sum_{\ell=0}^\infty |\widehat{\phi}(\ell)| \ell^{d+2k-1} < \infty$ then $\phi \in C^k[-1, 1]$.*

Proof. From [15, Equation (4.7.14)], observe that the derivative of a Gegenbauer polynomial satisfies $\frac{d}{dt} P_{\ell+1}^{(\lambda_d)}(t) = 2\lambda_d P_\ell^{(\lambda_d+2)}(t)$. Hence, for $k \leq \ell$,

$$\frac{d^k}{dt^k} P_\ell^{(\lambda_d)}(t) = 2^k \lambda_d \lambda_{d+2} \cdots \lambda_{d+2k-2} P_{\ell-k}^{(\lambda_d+2k)}(t),$$

while for $\ell < k$, the polynomial $P_\ell^{(\lambda_d)}$ is of degree at most $k - 1$ and is annihilated by $\frac{d^k}{dt^k}$. Since $\omega_{d+2} \lambda_{d+2} = \pi \omega_d$, it follows that

$$\left(\frac{\lambda_d + \ell}{\lambda_d \omega_d} \right) \frac{d^k}{dt^k} P_\ell^{(\lambda_d)}(t) = 2(2\pi)^{k-1} \left(\frac{\lambda_d + \ell}{\omega_{d+2k-2}} \right) P_{\ell-k}^{(\lambda_d+2k)}(t).$$

Utilizing a uniform bound on Gegenbauer polynomials ([15, Theorem 7.33.1]), $\max_{-1 \leq t \leq 1} |P_\ell^{(\lambda)}(t)| = \binom{\ell+2\lambda-1}{\ell}$, when $\lambda \geq 0$ we see that

$$\begin{aligned} \left| \left(\frac{\lambda_d + \ell}{\lambda_d \omega_d} \right) \frac{d^k}{dt^k} P_\ell^{(\lambda_d)}(t) \right| &\leq 2(2\pi)^{k-1} \left(\frac{\lambda_d + \ell}{\omega_{d+2k-2}} \right) \binom{\ell + d + k - 2}{\ell - k} \\ &\leq C_{d,k} \ell^{d+2k-1}. \end{aligned}$$

The result follows because the series $\sum_{\ell=0}^{\infty} \frac{\lambda_d + \ell}{\omega_d \lambda_d} \hat{\phi}(\ell) \frac{d^k}{dt^k} P_\ell^{(\lambda_d)}(t)$, is absolutely convergent, and, hence, equals $\frac{d^k}{dt^k} \phi(t)$. \square

When $1 \leq p < \infty$, the smoothness spaces we consider are the Sobolev (for integer smoothness) and Besov classes which we denote by $W_p^k(\mathbb{S}^d)$ and $B_{p,\infty}^s(\mathbb{S}^d)$, respectively. For $p = \infty$, we consider $C^k(\mathbb{S}^d)$ and the Besov classes $B_{\infty,\infty}^s(\mathbb{S}^d)$. These can be defined on \mathbb{S}^d in several, equivalent, customary ways. The simplest way to define Sobolev spaces is to use a partition of unity and local changes of variables to import the definition from \mathbb{R}^d as in [10, Sect. 3]. See the reference [16] for this and other definitions. Of principal importance to us is the fact that $L_p(\mathbb{S}^d) = W_p^0(\mathbb{S}^d)$ and Δ boundedly maps $W_p^s(\mathbb{S}^d)$ to $W_p^{s-2}(\mathbb{S}^d)$ (for $s \geq 2$). We postpone the discussion of Besov spaces until Section 6.

3. POLYHARMONIC KERNELS AND SURFACE SPLINES

The kernels we introduce in this section, the polyharmonic kernels, are fundamental solutions for certain elementary partial differential operators. In Section 3.1, we begin by defining the kernels in terms of the operators they invert. This indirect approach is taken because it is key to understanding the approximation scheme discussed in subsequent sections. A more direct expression in terms of Gegenbauer polynomials, (3), is also given.

Lemma 3.5 provides an asymptotic expansion $G_m \sim \sum_{j=0}^{\infty} \gamma_j \phi_{s+j}$ of polyharmonic kernels in terms of simpler kernels, called surface splines. This is developed in Section 3.3. In the course of demonstrating the asymptotic expansion, we make the complementary observation, Lemma 3.4, that the surface splines are polyharmonic kernels. This is the focus of Section 3.2.

3.1. Polyharmonic Kernels.

Definition 3.1 (Polyharmonic Kernels). *Let $m > d/2$ be an integer. For $r_1, \dots, r_m \in \mathbb{C}$, the polyharmonic kernel $G_m = G(\cdot; r_1, \dots, r_m)$, defined on $[-1, 1)$, is the fundamental solution for the product of perturbed Laplace – Beltrami operators $(\Delta - r_1) \dots (\Delta - r_m)$.*

Our interest in polyharmonic kernels stems from certain integral identities they satisfy. Such identities may hold for a general kernel \mathbf{k}_M (not necessarily polyharmonic, or even zonal),

$$(1) \quad f(x) = \int_{\mathbb{S}^d} \mathcal{L}_M(f - p_f)(\alpha) \mathbf{k}_M(x \cdot \alpha) d\alpha + p_f(x).$$

where \mathcal{L}_M is a differential operator of order M whose nullspace is contained in the finite dimensional space $\Pi_{\mathcal{J}} = \sum_{j \in \mathcal{J}} \mathcal{H}_j$ of spherical harmonics of prescribed degrees $j \in \mathcal{J}$, and where $p_f = \sum_{j \in \mathcal{J}} \sum_{m=1}^{N(d,j)} \langle f, Y_{j,m} \rangle Y_{j,m}$ is the (L_2) orthogonal projection onto this space. Because $\Pi_{\mathcal{J}}$ is finite dimensional, $\|p_f\|_X \leq \text{const}(X, p, \mathcal{J}) \|f\|_p$ for any norm $\|\cdot\|_X$. Hence, the identity (1) extends, by continuity, to every space $W_p^M(\mathbb{S}^d)$, with $1 \leq p < \infty$.

Definition 3.2. *If (1) holds for all $f \in C^M(\mathbb{S}^d)$, then \mathbf{k}_M is said to satisfy an integral identity of order M .*

When $\mathbf{k}_{2m} = G(\cdot; r_1, \dots, r_m)$, the operator is $\mathcal{L}_{2m} = (\Delta - r_1) \dots (\Delta - r_m)$, and \mathcal{J} must at least capture the indices corresponding to the eigenvalues used to construct \mathcal{L}_{2m} . That is, \mathcal{J} contains each index $j \in \mathbb{N}$ for which there is r_ℓ of the form $r_\ell = j(j+d-1)$ (there will be at most m such indices, although the set \mathcal{J} is free to contain more). Thus every kernel $G(\cdot; r_1, \dots, r_m)$ satisfies an integral identity of order $2m$.

We now show that the polyharmonic kernels can be decomposed as linear combinations of *surface splines* (perhaps more accurately called “restricted surface splines”), which are zonal functions

$$\phi_s(t) := \begin{cases} (1-t)^s \log(1-t) & \text{for } s \in \mathbb{N}; \\ (1-t)^s & s \in \mathbb{N} - \frac{1}{2} \end{cases}$$

Roughly, these are restrictions to the sphere of a well known family of RBFs: the surface splines, $|\cdot|^\beta$ and $|\cdot|^\beta \log|\cdot|$, produce the fundamental solution of the $(\beta+d)/2$ -fold Laplacian in \mathbb{R}^d . The zonal kernels considered here are restrictions of such to the sphere, by way of the identity $\frac{1}{2}(x-\alpha)^2 = 1 - x \cdot \alpha$. Providing this decomposition is important to determining error estimates, because there are precise bounds for the surface splines and their derivatives, especially near the singularity $x = \alpha$. For $t \geq 0$ it is not difficult to see that there exist constants $\beta_{s,j}$ so that

$$(2) \quad \left| \phi_s^{(j)}(t) \right| = \beta_{s,j} (1-t)^{s-j} \quad \text{for } j > s.$$

Our investigation of polyharmonic kernels begins with observing their expansions in Gegenbauer polynomials. The series expansion for G_m follows by Fourier inversion; its Fourier coefficients are obtained by reciprocating the symbol of the differential operator that G_m inverts:

$$G_m(x \cdot \alpha) = \sum_{\ell=1}^{\infty} \frac{\ell + \lambda_d}{\omega_d \lambda_d} \prod_{j=1}^m [\ell(\ell+d-1) - r_j]^{-1} P_\ell^{(\lambda_d)}(x \cdot \alpha).$$

It is often useful to adopt the notation $\vec{\ell} := \vec{\ell}(\ell, d) := \ell + \lambda_d$, in which case the Gegenbauer expansion becomes

$$(3) \quad G_m(x \cdot \alpha) = \sum_{\ell=1}^{\infty} \frac{\ell + \lambda_d}{\omega_d \lambda_d} \left[\prod_{j=1}^m \frac{1}{[\vec{\ell}^2 - \lambda_d^2] - r_j} \right] P_\ell^{(\lambda_d)}(x \cdot \alpha).$$

3.2. Surface Splines. The series expansion for surface splines is more difficult. It has been studied recently in [1] and [14]. These results allow a precise expansion of the kernel in Gegenbauer polynomials.

Lemma 3.3. *For $s \in \mathbb{N}/2$ satisfying $m := s + d/2 \in \mathbb{N}$, and $\vec{\ell} = \ell + \lambda_d$, there is a nonzero constant C_s (depending on s and d) such that the Fourier coefficient is*

$$\widehat{\phi_s}(\ell) = C_s \prod_{\nu=1}^m [\vec{\ell}^2 - (\nu - \frac{1}{2})^2]^{-1}$$

for $\ell > s$ when d is even, and for all ℓ when d is odd.

Proof. The formula $\phi_s(x \cdot \alpha) = \sum_{\ell=0}^{\infty} a_{\ell} P_{\ell}^{(\lambda_d)}(x \cdot \alpha)$ holds with

$$a_{\ell} = C_s \frac{\ell + \lambda_d}{\omega_d \lambda_d} \frac{\Gamma(\ell - s)}{\Gamma(s + \ell + d)}$$

for $\ell > s$ when $s \in \mathbb{N}$ by [1][(2.20)] and for all ℓ when $s \in \mathbb{N} - 1/2$ by [1][(2.12)]. Utilizing the notation $\vec{\ell} = \ell + \lambda_d$ and $\vec{s} = s + \lambda_d$ (and noting that \vec{s} is in $\mathbb{N} - 1/2$, since we assume that $s + d/2$ is an integer), the factor $\Gamma(\ell - s)/\Gamma(s + \ell + d)$ simplifies to $[(\vec{\ell}^2 - (\frac{1}{2})^2) \cdots (\vec{\ell}^2 - \vec{s}^2)]^{-1}$, and the lemma follows. \square

Thus, for any positive half-integer s , we have the expansion for surface splines: $\phi_s(x \cdot \alpha) = p(x \cdot \alpha) + C_s \sum_{\ell=0}^{\infty} \frac{\ell + \lambda_d}{\omega_d \lambda_d} \prod_{\nu=1}^m [\vec{\ell}^2 - (\nu - \frac{1}{2})^2]^{-1} P_{\ell}^{(\lambda_d)}(x \cdot \alpha)$, although the extra polynomial term $p \in \Pi_s[-1, 1]$ is only needed when d is even.

Lemma 3.4. *Let $m = s + d/2$. The kernel $(x, \alpha) \mapsto \phi_s(x \cdot \alpha)$ satisfies an integral identity of order $2m$ with operator*

$$\mathcal{L}_{2m} = \prod_{j=1}^m [\Delta - (j - d/2)(j + d/2 - 1)],$$

and $p_f = \sum_{\ell \leq s} \sum_{m=1}^{N(d, \ell)} \langle f, Y_{\ell, m} \rangle Y_{\ell, m}$, the projection onto Π_{2m-d} .

Proof. Since the symbol of the Laplacian is $\vec{\ell}^2 - \lambda_d^2$, the ν^{th} factor in the denominator of the Gegenbauer coefficient of ϕ_s is

$$\vec{\ell}^2 - (\nu - \frac{1}{2})^2 = \vec{\ell}^2 - \lambda_d^2 - [(\nu - \frac{1}{2})^2 - \lambda_d^2] = \vec{\ell}^2 - \lambda_d^2 - (\nu - \frac{d}{2})(\nu + \frac{d}{2} - 1).$$

Thus, when $d/2$ is fractional, ϕ_s is the fundamental solution for the invertible differential operator whose symbol is $\prod_{j=1}^m [(\vec{\ell}^2 - \lambda_d^2) - (j - \frac{d}{2})(j + \frac{d}{2} - 1)]$, since the eigenvalues of Δ are integers (the integers $k(k + d - 1)$). When $d/2$ is integral, the differential operator is invertible on the complement of the space of spherical harmonics of degree less than or equal to s . \square

3.3. Surface Spline Expansion of Polyharmonic Kernels.

Lemma 3.5. *For positive integers m and d , let $s = m - d/2$. The polyharmonic kernel $(x, \alpha) \mapsto G_m(x \cdot \alpha; r_1, \dots, r_m)$ can be written as*

$$G_m(x \cdot \alpha) = \sum_{j=0}^{J-1} \gamma_j \phi_{s+j}(x \cdot \alpha) + R_J(x \cdot \alpha)$$

with $R_J \in C^{(J+s-\epsilon)}([-1, 1])$.

Proof. We begin by expanding each of the Fourier coefficients of G_m . From (3) we observe that $\widehat{G_m}(\ell) = \prod_{j=1}^m (\vec{\ell}^2 - \lambda_d^2 - r_j)^{-1}$. Factoring $\vec{\ell}^{-2m}$, we have, for $\ell > \max(\sqrt{|r_1|}, \dots, \sqrt{|r_m|})$, that

$$\widehat{G_m}(\ell) = \vec{\ell}^{-2m} \prod_{j=1}^m \left(1 - \frac{\lambda_d^2 + r_j}{\vec{\ell}^2} \right)^{-1} = \vec{\ell}^{-2m} \left(1 + \sum_{n=1}^{\infty} A_n \vec{\ell}^{-2n} \right).$$

The second equality follow by writing each factor in the product as a Neumann series (i.e., a series of the form $(1 - a)^{-1} = \sum_{j=0}^{\infty} a^j$), and then by multiplying the m series. We do likewise for the coefficients of ϕ_{s+j} (determined in Lemma 3.3) when $\ell > s + J$:

$$\widehat{\phi_{s+j}}(\ell) = C_{s+j} \vec{\ell}^{-2(m+j)} \left(1 + \sum_{n=1}^{\infty} B_{j+n,j} \vec{\ell}^{-2n} \right).$$

This allows us to choose the coefficients $\gamma_0, \gamma_1, \dots$ in succession, via $\gamma_j := C_{s+j}^{-1} (A_j - \sum_{k=0}^{j-1} \gamma_k B_{j,k})$. With this choice, the first J terms in the asymptotic expansion of $\widehat{R_J}(\ell)$ are forced to vanish. The fact that each γ_j depends only on the previous coefficients γ_k , $k < j$, is evident from Table 1. The coefficients of the remainder term are determined to be

$$\begin{aligned} \widehat{G_m}(\ell) &= \vec{\ell}^{-2m} (1 + A_1 \vec{\ell}^{-2} + A_2 \vec{\ell}^{-4} + A_3 \vec{\ell}^{-6} + \dots) \\ \widehat{\phi_s}(\ell) &= C_s \vec{\ell}^{-2m} (1 + B_{1,0} \vec{\ell}^{-2} + B_{2,0} \vec{\ell}^{-4} + B_{3,0} \vec{\ell}^{-6} + \dots) \\ \widehat{\phi_{s+1}}(\ell) &= C_{s+1} \vec{\ell}^{-2m} (\vec{\ell}^{-2} + B_{2,1} \vec{\ell}^{-4} + B_{3,1} \vec{\ell}^{-6} + \dots) \\ \widehat{\phi_{s+2}}(\ell) &= C_{s+2} \vec{\ell}^{-2m} (\vec{\ell}^{-4} + B_{3,2} \vec{\ell}^{-6} + \dots) \\ &\dots \end{aligned}$$

TABLE 1. Expansion of Gegenbauer coefficients

$$\widehat{R_J}(\ell) = \widehat{G_m}(\ell) - \sum_{j=0}^{J-1} \gamma_j.$$

$$|\widehat{R_J}(\ell)| \leq \sum_{k=0}^{\infty} \left| A_{J+k} - \sum_{j=0}^{J-1} \gamma_j B_{J+k-j-1,j} \right| \vec{\ell}^{-2(m+J+k)} \leq \text{const}(J) < \infty$$

for sufficiently large $\vec{\ell}$. By Proposition 2.1, the lemma follows. \square

4. REPLACING THE KERNELS I: FINDING THE COEFFICIENTS

We now wish to investigate a ‘coefficient kernel’ $\mathbf{a} : \Xi \times \mathbb{S}^d \rightarrow \mathbb{R}$ that will allow us to effectively replace a $\mathbf{k}(x \cdot \alpha)$ with $\sum_{\xi \in \Xi} \mathbf{a}(\xi, \alpha) \mathbf{k}(x \cdot \xi)$ in the representation (1). To do so, the *exchange* $\mathbf{e}_{\mathbf{k}}$ given by:

$$\mathbf{e}_{\mathbf{k}}(x, \alpha) := |\mathbf{k}(x \cdot \alpha) - \sum_{\xi \in \Xi} \mathbf{a}(\xi, \alpha) \mathbf{k}(x \cdot \xi)|$$

must be appropriately small in L_∞ , and it must decay away from $\alpha = x$. The remarkable thing is that this can be achieved using only a fixed number of centers near to the singularity. In this section, we develop a technique for choosing coefficients $\mathbf{a}(\xi, \alpha)$ that – in the following section – is shown to provide an appropriately small exchange.

The two key quantities we need to resolve are the spherical harmonic *precision* (the degree of spherical harmonics reproduced by the coefficient kernel) and the rate of decay of the error as $\text{dist}(x, \alpha)$ increases. As in the Euclidean setting, these are related: the higher the degree of spherical harmonic precision, the more rapidly the exchange decays away from the singularity.

Definition 4.1 (CKC). *For a set of centers $\Xi \subset \mathbb{S}^d$ the kernel $\mathbf{a} : \Xi \times \mathbb{S}^d \rightarrow \mathbb{R}$ satisfies the Coefficient Kernel Conditions (or CKC) with precision L , radius ρ and stability K if it is measurable and the following three conditions hold:*

CKC 1 (Support): $\mathbf{a}(\xi, \alpha) = 0$ when $\text{dist}(\xi, \alpha) > \rho$.

CKC 2 (Precision): For $S \in \Pi_L$, $\sum_{\xi \in \Xi} \mathbf{a}(\xi, \alpha) S(\xi) = S(\alpha)$

CKC 3 (Stability): $\max_{\alpha \in \mathbb{S}^d} \sum_{\xi \in \Xi} |\mathbf{a}(\xi, \alpha)| \leq K$.

Such a local reproduction property always holds for sufficiently dense centers. This is demonstrated in the following lemma.

Lemma 4.2. *Given a precision L and centers Ξ having density $h < h_0$ (with h_0 determined by L), there exists a coefficient kernel $\mathbf{a} : \Xi \times \mathbb{S}^d \rightarrow \mathbb{R}$ satisfying the CKC with radius $\rho = 48L^2h$ and stability $K = 2$.*

Proof. Let $\Xi_\alpha := \Xi \cap C(\alpha, \rho)$, the set of centers a distance ρ from α . Following what is, by now, a fairly standard technique in scattered data approximation (originally developed for the sphere in [9], and deftly exposted in [19, Ch. 3]), a coefficient kernel is shown to exist if the sampling operator

$$R_\alpha : \Pi_L \rightarrow (\Pi_L)_{|\Xi_\alpha} : p \mapsto p|_{\Xi_\alpha}$$

is boundedly invertible when the domain and range are endowed with the L_∞ and ℓ_∞ topologies, respectively. To be precise, we must show that the norm of the inverse of the sampling operator is bounded by 2: $\|R_\alpha^{-1}\| \leq 2$, which is accomplished in Lemma 4.3, below. Bounded invertibility of R_α implies that the norm of the adjoint

$$(R_\alpha^{-1})' : \Pi' \rightarrow \left((\Pi_L)_{|\Xi_\alpha} \right)'$$

is similarly bounded. By the Hahn-Banach theorem, there is a norm-bounded extension of the functional $(R_\alpha^{-1})' \delta_\alpha \in ((\Pi_L)_{|\Xi_\alpha})'$ in the space, $(\ell_\infty(\Xi_\alpha))'$. This can be viewed as an element of $\ell_1(\Xi_\alpha)$, and, by zero extension, it is in $\ell_1(\Xi)$. We call this sequence $\mathbf{a}(\cdot, \alpha)$ and note that its ℓ_1 norm is bounded by $\|(R_\alpha^{-1})'\| \|\delta_\alpha\| \leq 2$.

The measurability of the kernel \mathbf{a} is a consequence of its piecewise continuity, which we now demonstrate. For each $v \subset \Xi$, we define the open set $\Omega_v := \cap_{\xi \in v} C(\xi, \rho)$. These can be refined to a (finite) collection of sets

$$\tilde{\Omega}_v := \Omega_v \setminus \bigcup_{v \subsetneq \zeta} \Omega_\zeta$$

that partitions \mathbb{S}^d . For each α in $\tilde{\Omega}_v$, the sampling operators R_α share a common target $\Pi|_v$, and the operator valued map $\alpha \mapsto R_\alpha$ is well defined and Lipschitz. Indeed, $\|R_\alpha p - R_{\alpha'} p\| \leq C|\alpha - \alpha'| \|\nabla p\|_\infty \leq C_L |\alpha - \alpha'| \|p\|_{L_\infty}$ implies that $\|R_\alpha - R_{\alpha'}\| \leq C_L |\alpha - \alpha'|$. The inverse is similarly Lipschitz, because $R_\alpha^{-1} - R_{\alpha'}^{-1} = R_\alpha^{-1} [R_{\alpha'} - R_\alpha] R_{\alpha'}^{-1}$. For $\alpha \in \tilde{\Omega}_v$ the family of sequences $\mathbf{a}(\cdot, \alpha)$ have their support in v . To show that $\alpha \mapsto \mathbf{a}(\cdot, \alpha)$ is continuous we simply observe that

$$\begin{aligned} \|\mathbf{a}(\cdot, \alpha) - \mathbf{a}(\cdot, \alpha')\|_{\ell_1} &\leq \|R_\alpha^{-1} - R_{\alpha'}^{-1}\| \|\delta_\alpha\| + \|(R_{\alpha'}^{-1})'\| \|\delta_{\alpha'} - \delta_\alpha\| \\ &= O(|\alpha - \alpha'|). \end{aligned}$$

□

Lemma 4.3. *Given a precision L and centers Ξ with density $h < h_0$, let $\rho = 48L^2$ and $\Xi_\alpha := \Xi \cap C(\alpha, \rho)$ for each $\alpha \in \mathbb{S}^d$. The sampling operator R_α is boundedly invertible on the space of spherical harmonics of degree L or less, and*

$$\|p\|_{L_\infty(C(\alpha, \rho))} \leq 2 \|R_\alpha p\|_{\ell_\infty(\Xi_\alpha)}.$$

Proof. This is accomplished by noting that spherical harmonics, when restricted to great circles, are trigonometric polynomials. From this we can apply the Markov inequality of Videnskii[17], which states that for a trigonometric polynomial, τ of degree n

$$|\tau'(\theta)| \leq 2n^2 \cot(\omega/2) \|\tau\|_{L_\infty(-\omega, \omega)} \quad \text{for } \omega < \pi, |\theta| \leq \omega$$

to control the size of a spherical harmonics having many zeros in a spherical cap.

Select $p \in \Pi_L$ and find x_0 such that $|p(x_0)| = \|p\|_{L_\infty(C(\alpha, \rho))}$. Following Wendland [19, p.30], we take $\xi \in \Xi \cap C(\alpha, \rho)$ so that ξ is in a cone with vertex x_0 and distance from x_0 less than $h + \frac{h}{\sin \theta} \leq 3h$ (this is possible because a cap of radius h with center located at a distance $\frac{h}{\sin \theta}$ from x_0 is contained in the cone of aperture θ). Let \hat{x}_0 be the terminal point of the geodesic segment starting at x_0 , passing through ξ and having length ρ . Restricting p to this geodesic gives a trigonometric polynomial of degree L . Vis., there

is $q : [-\pi, \pi] \rightarrow \mathbb{C}$, $q(\theta) = \sum_{|j| \leq 2m} a_j e^{ij\theta}$, such that $q(-\rho/2) = p(x_0)$ and $q(\rho/2) = p(\hat{x}_0)$ and

$$|p(x_0) - p(\xi)| \leq \int_{-\rho/2}^{-\rho/2 + |x_0 - \xi|} |q'(t)| dt.$$

By Videnskii's Markov inequality, $|q'(t)| \leq 2L^2 \cot(\rho/4) \|q\|_{L_\infty(-\rho/2, \rho/2)}$, and, consequently, we have that

$$|p(x_0) - p(\xi)| \leq 3h \, 2(L)^2 \frac{4}{\rho} \|p\|_{L_\infty(C(\alpha, \rho))} \leq \frac{1}{2} \|p\|_{L_\infty(C(\alpha, \rho))}.$$

Thus $\|p\|_{L_\infty(C(\alpha, \rho))} \leq 2\|p|_\Xi\|_{\ell_\infty}$ and the lemma is proved. \square

A consequence of the CKC is that for any x and any zonal function \mathbf{k} that is smooth on the interval $\mathcal{J}_x := [\min Q_x, \max Q_x]$, where $Q_x = \{x \cdot \alpha\} \cup \{x \cdot \xi : \xi \in (\Xi \cap C(\alpha, \rho))\}$, the exchange can be estimated in terms of the length of the interval \mathcal{J}_x and the size of derivatives of \mathbf{k} *purely on* \mathcal{J}_x . This is the point of the following lemma:

Lemma 4.4. *Given a coefficient kernel satisfying the CKC with precision L , if $\mathbf{k} \in C^{(L+1)}(\mathcal{J}_x)$, then the exchange satisfies*

$$(4) \quad \mathbf{e}_{\mathbf{k}}(x, \alpha) \leq \frac{\|\mathbf{a}(\cdot, \alpha)\|_{\ell_1}}{L!} \max_{\xi \in \Xi \cap C(\alpha, \rho)} |x \cdot (\alpha - \xi)|^{L+1} \max_{t \in \mathcal{J}_x} |\mathbf{k}^{(L+1)}(t)|$$

Proof. Let both x and α be fixed, set $x \cdot \alpha = t_\alpha \in [-1, 1]$ and choose the Taylor polynomial of degree L , q_{L, t_α} , of \mathbf{k} expanded about t_α . Now q_{L, t_α} may be rewritten as a linear combination of Gegenbauer polynomials,

$$q_{L, t_\alpha}(t) = \sum_{\ell=0}^L \widehat{q_{L, t_\alpha}}(\ell) \frac{\ell + \lambda_d}{\omega_d \lambda_d} P_\ell^{(\lambda_d)}(t).$$

Note, furthermore, that $q_{L, t_\alpha}(x \cdot \alpha) - \sum \mathbf{a}(\xi, \alpha) q_{L, t_\alpha}(x \cdot \xi) = 0$ by the addition theorem, since $q_{L, t}(x \cdot \zeta) = \sum_\ell \widehat{q_{L, t}}(\ell) \sum_m Y_{\ell, m}(x) Y_{\ell, m}(\zeta)$ and each $Y_{\ell, m}(\zeta)$ is annihilated by $\mu = \delta_\alpha - \sum \mathbf{a}(\xi, \alpha) \delta_\xi$. Consequently, $\mathbf{e}_{\mathbf{k}}(x, \alpha) \leq \sum_\xi |\mathbf{a}(\xi, \alpha)| |\mathbf{k}(x \cdot \xi) - q_{L, t_\alpha}(x \cdot \xi)|$, and the Taylor's theorem gives:

$$|\mathbf{k}(t_\xi) - q_{L, t_\alpha}(t_\xi)| \leq \frac{1}{L!} |t_\xi - t_\alpha|^{L+1} \max_{u \in \text{co}(t_\xi, t_\alpha)} |\mathbf{k}^{(L+1)}(u)|.$$

\square

5. REPLACING THE KERNELS II: ESTIMATES

Having found coefficients suitable for replacing the kernel in a representation (1), we now obtain estimates on the exchange in an effort to estimate the norm of the operator $E_{\mathbf{k}} : L_p(\mathbb{S}^d) \rightarrow L_p(\mathbb{S}^d)$, defined by $E_{\mathbf{k}} g(x) = \int \mathbf{e}_{\mathbf{k}}(x, \alpha) g(\alpha) d\alpha$. The bound, $\|E_{\mathbf{k}}\| := \sup_{0 \neq g \in L_p} \|E_{\mathbf{k}} g\|_p / \|g\|_p$

gives us essentially the error estimates we desire, since the pointwise error, $\mathcal{E}(x) := |f(x) - \int_{\mathbb{S}^d} \mathbf{a}(\xi, \alpha) \mathbf{k}(x \cdot \xi) \mathcal{L}_{2m} f(\alpha) d\alpha|$, satisfies, by (1),

$$\begin{aligned} \mathcal{E}(x) &= \left| \int_{\mathbb{S}^d} \left(\mathbf{k}(x \cdot \alpha) - \sum \mathbf{a}(\xi, \alpha) \mathbf{k}(x \cdot \xi) \right) \mathcal{L}_{2m} f(\alpha) d\alpha \right| \\ &\leq \int_{\mathbb{S}^d} \mathbf{e}_{\mathbf{k}}(x, \alpha) |\mathcal{L}_{2m} f(\alpha)| d\alpha. \end{aligned}$$

In other words, $\|\mathcal{E}\|_p \leq \|E_{\mathbf{k}} \mathcal{L}_{2m} f\|_p \leq \|E_{\mathbf{k}}\| \|\mathcal{L}_{2m} f\|_p$. Because of the expansion from Lemma 3.5, we focus on obtaining the estimates for surface splines first, before moving to polyharmonic functions in general.

Lemma 5.1. *Let $m = s + d/2$. Assume \mathbf{a} is a coefficient kernel satisfying the CKC with radius ρ , precision $2m$ and stability K . Then for $j \in \mathbb{N}$ the exchange of the kernel ϕ_{s+j} satisfies*

$$\mathbf{e}_{\phi_{s+j}}(x, \alpha) \leq \text{const}(K, m, j, d) \rho^{2(m+j)-d} \left(1 + \frac{\text{dist}(x, \alpha)}{\rho} \right)^{2j-d-1}$$

Proof. We consider three regions, for a fixed ‘north pole’ $\alpha \in \mathbb{S}^d$:

$\Omega_1 := \{x \mid \pi/2 < \text{dist}(x, \alpha) \leq \pi\}$, where the surface spline is smooth;
 $\Omega_2 := \{x \mid 0 < \text{dist}(x, \alpha) \leq \rho\}$, a cap of radius ρ near the north pole;
 $\Omega_3 := \{x \mid \rho < \text{dist}(x, \alpha) \leq \pi/2\}$: a band where high order derivatives decay.

Ω_1 : We note that outside the spherical cap $C(\alpha, \pi/2)$ the $(2m+1)^{\text{st}}$ derivatives of ϕ_{s+j} are bounded by $\beta_{s+j, 2m+1}$, and we can use (4) to obtain $\mathbf{e}_{\phi_{s+j}}(x, \alpha) \leq \rho^{2m+1} \frac{K}{2m!} \beta_{s+j, 2m+1}$.

Ω_2 : In the cap nearest to α , we use the fact that ϕ_{s+j} has a high order zero. Here we need a relationship (used later, as well) between the geodesic distance of two points and their inner product

$$(5) \quad 1 - \frac{1}{2} (\text{dist}(x, \zeta))^2 \leq x \cdot \zeta \leq 1 - \frac{4}{\pi^2} (\text{dist}(x, \zeta))^2 \quad \text{for } \text{dist}(x, \zeta) \leq \pi/2.$$

For even d , the proof is complicated by the log factor, so we consider this case only, as the odd case follows by a similar but much simpler argument. We proceed by writing

$$\begin{aligned} &|1 - x \cdot \zeta|^{s+j} \log |1 - x \cdot \zeta| \\ &= \rho^{2(s+j)} \left| \frac{1 - x \cdot \zeta}{\rho^2} \right|^{s+j} \log |\rho^2| + \rho^{2(s+j)} \left| \frac{1 - x \cdot \zeta}{\rho^2} \right|^{s+j} \log \left| \frac{1 - x \cdot \zeta}{\rho^2} \right| \end{aligned}$$

Since s is even when d is even, the first term is simply a spherical harmonic in ζ (by the addition theorem), of degree $s+j \leq 2m$, and is therefore annihilated by the functional $\mu = \delta_\alpha - \sum_{\xi \in \Xi} \mathbf{a}(\xi, \alpha) \delta_\xi$. Thus, we need only

apply μ to the second term; we obtain (by the left hand side of (5)),

$$\begin{aligned} e_{\phi_{s+j}}(x, \alpha) &\leq (1+K)\rho^{2(s+j)} \max_{\zeta \in C(\alpha, 3\rho)} \left| \frac{1-x \cdot \zeta}{\rho^2} \right|^{s+j} \log \left| \frac{1-x \cdot \zeta}{\rho^2} \right| \\ &\leq \rho^{2(s+j)}(1+K) \left[\max_{0 \leq t \leq 9/2} |t|^{s+j} \log |t| \right] \end{aligned}$$

Ω_3 : The estimate (5) bounds the derivatives of ϕ_{s+j} , but in the northern hemisphere, we can achieve better estimates for $|x \cdot (\xi - \alpha)|$, in the sense that this inner product becomes considerably smaller than ρ when x and α are close. Decompose $C(\alpha, \kappa) \setminus C(\alpha, 2\rho)$ *en annuli*, and note that

$$(6) \quad \text{dist}(x, \alpha) \leq 2^k \rho \quad \Rightarrow \quad |x \cdot (\alpha - \xi)| \leq 2^{k+1} \rho^2.$$

For $2^{k-1} \rho \leq \text{dist}(x, \alpha) \leq 2^k \rho$ we estimate $\max \mathcal{J}_x \leq 1 - \frac{4}{\pi^2} (2^{k-1} \rho)^2$ by (5) to obtain bounds on the $(2m+1)^{\text{th}}$ derivatives of ϕ_{s+j} on \mathcal{J}_x . On the other hand, we can apply (6) to estimate $|x \cdot (\alpha - \xi)|$. Thus,

$$\begin{aligned} e_{\phi_{s+j}}(x, \alpha) &\leq \frac{K}{2m!} \beta_{s+j, 2m+1} (2^{k+1} \rho^2)^{2m+1} \left(\frac{4}{\pi^2} (2^{k-1} \rho)^2 \right)^{s+j-2m-1} \\ &= \text{const}(K, m, j, d) (2^k)^{-d-1+2j} \rho^{2s+2j} \end{aligned}$$

The first inequality follows from (4) using (2),(5) and (6) while the second inequality is a consequence of the fact that $d = 2m - 2s$. \square

6. MAIN RESULTS

We are now in a position to prove our main results, that polyharmonic kernels and surface splines deliver L_p approximation orders commensurate with the L_p smoothness of the target function, at least up to a (putative) ‘saturation order’: the order of the differential operator that the kernel inverts. We begin by giving ‘high order’ results, for functions of ‘full’ smoothness. Afterwards, we give the lower orders and the corresponding smoothness spaces by means of real interpolation.

Theorem 6.1. *Assume the coefficient kernel $\mathbf{a} : \Xi \times \mathbb{S}^d \rightarrow \mathbb{R}$ satisfies CKC with radius ρ , precision $2m$ and stability K . Assume, further, that the kernel \mathbf{k} provides an integral identity (1) of order $2m$ and can be decomposed as $\mathbf{k} = \sum_{j=0}^{2m-s} \gamma_j \phi_{s+j} + R$, with $s = m - d/2$ and remainder $R \in C^{(2m)}[-1, 1]$. Then for $f \in W_p^{2m}(\mathbb{S}^d)$, if $1 \leq p < \infty$, or for $f \in C^{2m}(\mathbb{S}^d)$ when $p = \infty$, the approximant*

$$T_{\Xi} f(x) = p_f(x) + \sum_{\xi \in \Xi} A_{\xi} \mathbf{k}(x \cdot \xi),$$

with coefficients $A_{\xi} = \int_{\mathbb{S}^d} \mathcal{L}_{2m}(f - p_f)(\alpha) \mathbf{a}(\xi, \alpha) d\alpha$, $\xi \in \Xi$, converges to f in $L_p(\mathbb{S}^d)$ with error:

$$\|f - T_{\Xi} f\|_{L_p(\mathbb{S}^d)} \leq \text{const}(K, \mathbf{k}) \rho^{2m} \|f\|_{W_p^{2m}(\mathbb{S}^d)}$$

and with coefficients satisfying $\|A\|_{\ell_1(\Xi)} \leq \text{const} (K, \mathbf{k}) \|f\|_{W_p^{2m}(\mathbb{S}^d)}$.

The decomposition, $\mathbf{k} = \sum_{j=0}^{2m-s} \gamma_j \phi_{s+j} + R$, means that this result holds for surface splines themselves, and by Lemma 3.5 it holds for polyharmonic kernels G_m as well.

Proof. We begin by estimating the operator norm of $E_{\mathbf{k}}$. To do this for $1 \leq p \leq \infty$, we simply find estimates for $\|E_{\mathbf{k}}\|_{1 \rightarrow 1}$ and $\|E_{\mathbf{k}}\|_{\infty \rightarrow \infty}$, obtaining the $\|E_{\mathbf{k}}\|_{p \rightarrow p}$ norm by interpolation. By symmetry, both the L_1 and L_∞ operator norms are bounded by $\sup_{\alpha \in \mathbb{S}^d} \int_{\mathbb{S}^d} |e_{\mathbf{k}}(x, \alpha)| dx$.

The decomposition of the kernel permits us to estimate this integral as the sum of the constituent integrals $\int |e_{\phi_{s+j}}(x, \alpha)| dx$, for $j = 0 \dots 2m - s$ and $\int |e_R(x, \alpha)| dx$. The latter can be estimated using Lemma 4.4 directly: $\int_x |e_R(x, \alpha)| dx \leq \frac{K}{2m!} \omega_d \rho^{2m} \|R^{(2m)}\|_{L_\infty[-1,1]}$. The integrals of the kernels $e_{\phi_{s+j}}(x, \alpha)$ are estimated by splitting the sphere into the southern hemisphere, Ω_1 , and northern hemisphere, Ω_1^c . By Lemma 5.1, $e_{\phi_{s+j}}$ is bounded uniformly over Ω_1 by $C\rho^{2m+1}$, so $\int_{\Omega_1} |e_{\phi_{s+j}}(x, \alpha)| d\alpha \leq C\rho^{2m+1}$.

On Ω_1^c we integrate using polar coordinates, obtaining:

$$\begin{aligned} \int_{\Omega_1^c} |e_{\phi_{s+j}}(x, \alpha)| d\alpha &\leq C \int_{\Omega_1^c} \rho^{2(m+j)-d} \left(1 + \frac{\text{dist}(x, \alpha)}{\rho}\right)^{2j-d-1} d\alpha \\ &\leq C\rho^{2(m+j)} \left(1 + \int_1^{\pi/(2\rho)} R^{2j-2} dR\right) \\ &\leq C\rho^{2(m+j)} \left(1 + \left(\frac{\pi}{2}\right)^{2j-1} \rho^{1-2j}\right) \leq C\rho^{2m}. \end{aligned}$$

To bound the coefficients, we make the estimate

$$\sum_{\xi \in \Xi} |A_\xi| \leq \sum_{\xi \in \Xi} \int |\mathbf{a}(\alpha, \xi)| |\mathcal{L}_{2m}(f - p_f)(\alpha)| d\alpha.$$

This is less than

$$\begin{aligned} \int \sum_{\xi \in \Xi} |\mathbf{a}(\alpha, \xi)| |\mathcal{L}_{2m}(f - p_f)(\alpha)| d\alpha &\leq K \int_{\mathbb{S}^d} |\mathcal{L}_{2m}(f - p_f)(\alpha)| d\alpha \\ &\leq CK \|f\|_{W_p^{2m}}. \end{aligned}$$

□

The previous theorem requires the target function to have $2m$ derivatives in L_p , which is quite restrictive. To treat more general functions, we can first approximate a target function of lower smoothness by a nearby member, g , of W_p^{2m} , and apply the theorem to g instead of f . This is an old trick in approximation theory, and it is a consequence of the fact that the Besov spaces are interpolation spaces of Sobolev spaces. We make use of the Besov

spaces $B_{p,\infty}^\sigma$, $1 \leq p < \infty$ and $0 < \sigma < 2m$, which are the spaces of L_p functions with norm

$$\|f\|_{B_{p,\infty}^\sigma(\mathbb{S}^d)} := \sup_{t>0} \left(t^{-\frac{\sigma}{2m}} \inf \left\{ \|f - g\|_p + t\|g\|_{W_p^{2m}} : g \in W_p^{2m}(\mathbb{S}^d) \right\} \right).$$

When $p = \infty$, the norm can be rewritten with C^{2m} replacing W_p^{2m} . Rather than paraphrase the theory here, we point the interested reader to [16, Chapters 1 and 7] for the pertinent theorems and definitions.

Corollary 6.2. *In the setting of the previous theorem, if $f \in B_{p,\infty}^\sigma(\mathbb{S}^d)$ for $1 \leq p \leq \infty$ with $0 < \sigma < 2m$ then $\text{dist}(f, S(\mathbf{k}, \Xi))_p \leq \text{const } \rho^\sigma \|f\|_{B_{p,\infty}^\sigma}$, and this can be accomplished with an approximant*

$$s_{\xi,f}(x) = \sum_{\xi \in \Xi} A_\xi \mathbf{k}(x \cdot \xi) + p(\xi),$$

with $p \in \Pi_d$ and with coefficients satisfying $\|A\|_{\ell_1(\Xi)} \leq C\rho^{\sigma-2m}\|f\|_{B_{p,\infty}^\sigma(\mathbb{S}^d)}$.

Proof. By real interpolation, we have, for every $t > 0$, that $\inf\{\|f - g\|_p + t\|g\|_{W_p^{2m}} : g \in W_p^{2m}(\mathbb{S}^d)\} \leq t^{\frac{\sigma}{2m}}\|f\|_{B_{p,\infty}^\sigma(\mathbb{S}^d)}$. This implies, taking $t = \rho^{2m}$, that we can find $g_\rho \in W_p^{2m}(\mathbb{S}^d)$ satisfying

$$\begin{aligned} \|f - g_\rho\|_p &\leq 2\rho^\sigma \|f\|_{B_{p,\infty}^\sigma(\mathbb{S}^d)}; \\ \|g_\rho\|_{W_p^{2m}(\mathbb{S}^d)} &\leq 2\rho^{\sigma-2m} \|f\|_{B_{p,\infty}^\sigma(\mathbb{S}^d)} \end{aligned}$$

Applying the previous theorem to g_ρ gives $\|f - s_\Xi g\|_p \leq \|f - g\|_p + \|g - s_\Xi g\|_p \leq 2\rho^\sigma \|f\|_{B_{p,\infty}^\sigma(\mathbb{S}^d)} + \text{const } \rho^{2m} \rho^{\sigma-2m} \|f\|_{B_{p,\infty}^\sigma(\mathbb{S}^d)}$. The coefficient estimate follows by a similar argument. \square

By Lemma 4.2, we can apply the previous results to approximation with sufficiently dense centers.

Corollary 6.3. *For centers $\Xi \in \mathbb{S}^d$, having fill distance $h < h_0$, (with h_0 given by Lemma 4.2), if $f \in X_p^s$*

$$\text{dist}(f, S(\mathbf{k}, \Xi))_p \leq \text{const } (\mathbf{k}) h^s \|f\|_{X_p^s}$$

where X_p^s is W_p^{2m} when $s = 2m$, or $B_{p,\infty}^s$ when $0 < s < 2m$.

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